

# Crew's Euler Characteristic Formula Fails for Nonzero Slopes

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Let  $f : X \rightarrow Y$  be a finite étale cover of separated schemes of finite type over a field  $k$  of characteristic  $p > 0$ . If  $l \neq p$ , the Euler-Poincaré formula (a special case of Riemann-Hurwitz) states that  $\chi_c(X, \mathbb{Q}_l) = (\deg f)\chi(Y, \mathbb{Q}_l)$ , where  $\chi_c$  denotes the Euler characteristic

$$\chi_c(X, \mathbb{Q}_l) = \sum_{i=0}^{2 \dim X} \dim_{\mathbb{Q}_l} H_c^i(X, \mathbb{Q}_l)$$

with respect to étale cohomology with compact supports.

This relation fails in general for  $l = p$ , but holds in an important special case. Suppose  $f$  is a Galois cover and  $\deg f$  is a power of  $p$ . Then the relation  $\chi_c(X, \mathbb{Q}_p) = (\deg f)\chi(Y, \mathbb{Q}_p)$  holds; for complete curves, this is due to Shafarevich [2], for arbitrary curves it is equivalent to the Deuring-Shafarevich formula, and in general it is due to in dimension 1 and by Crew [1] in general.

For  $X$  and  $Y$  smooth and proper over  $\operatorname{Spec} k$ , this assertion can be reinterpreted in terms of crystalline cohomology. Namely, in this case, the crystalline cohomology groups  $H_{\text{crys}}^i(X/W)$  are finitely generated modules over the Witt ring  $W$  of  $k$  with a semilinear endomorphism  $F$  (the Frobenius). For any rational number  $\lambda$ , we let  $h_\lambda^i(X)$  be the multiplicity of the slope  $\lambda$  in  $H_{\text{crys}}^i(X/W)$ ; then  $h_0^i(X) = \dim H_c^i(X, \mathbb{Q}_l)$ . If we put

$$\chi_\lambda(X) = \sum_{i=0}^{2 \dim X} (-1)^i h_\lambda^i(X),$$

then Crew's theorem is that  $\chi_0(X) = |G|\chi_0(Y)$ .

Crew asks whether  $\chi_\lambda(X) = |G|\chi_\lambda(Y)$  for other values of  $\lambda$ . We present an explicit counterexample to this assertion with  $p = 2$  and  $\dim X = 1$ .

Let  $k = \overline{\mathbb{F}_2}$ . Consider the curves

$$\begin{aligned} C : u^2 - u &= \frac{1 + x^2 + x^8 + x^{14} + x^{18}}{x^{21}} \\ D : v^2 - v &= \frac{1}{x + 1} \\ Y : w^2 - w &= \frac{1 + x^2 + x^8 + x^{14} + x^{18}}{x^{21}} + \frac{1}{x + 1} \end{aligned}$$

and let  $X$  be the fibre product of  $C$  and  $D$  over the maps to  $\mathbb{P}_1$  given by  $x$ . Then  $X$  admits a map to  $Y$  given by setting  $w = u + v$ ; it is easily verified that this map is étale. Furthermore, using Riemann-Hurwitz, one calculates  $g(C) = 10$ ,  $g(D) = 0$ ,  $g(Y) = 11$ , and  $g(X) = 21$ .

The characteristic polynomials of Frobenius on  $H^1(C)$ ,  $H^1(X)$ ,  $H_1(Y)$  can be obtained by counting points over finite extensions of  $\mathbb{F}_2$ . We compute

$$\begin{aligned} P_C(t) &= 1 - 32t^{10} + 1024t^{20} \\ P_Y(t) &= 1 + t + 2t^2 + 4t^3 + 4t^4 + 4t^5 + 8t^6 + 8t^7 + 8t^8 + 16t^9 + 32t^{10} + 32t^{11} \\ &\quad + 64t^{12} + 64t^{13} + 64t^{14} + 128t^{15} + 256t^{16} + 256t^{17} + 512t^{18} + 1024t^{19} \\ &\quad + 1024t^{20} + 1024t^{21} + 2048t^{22} \end{aligned}$$

and  $P_X(t) = P_C(t)P_Y(t)$ ; of course  $P_D(t) = 1$ . From these polynomials we can read off the  $h_\lambda^i$ ; namely,

$$h_0^1(Y) = h_1^1(Y) = 1, h_{3/7}^1(Y) = h_{4/7}^1(Y) = 7, h_{1/2}^1(Y) = 6 \quad (1)$$

$$h_0^1(X) = h_1^1(X) = 1, h_{3/7}^1(X) = h_{4/7}^1(X) = 7, h_{1/2}^1(X) = 26 \quad (2)$$

and so  $\chi_\lambda(X) \neq 2\chi_\lambda(Y)$  for  $\lambda \in \{3/7, 1/2, 4/7\}$ .

The underlying phenomenon seems to be that while  $C$  is supersingular, the generic curve of the form  $u^2 - u = A(x)/x^{21}$  has slopes  $3/7$  and  $4/7$  with multiplicity 7. It appears that the supersingularity of  $C$  is unstable under fiber products.

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## References

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- [2] I. Shafarevich, On  $p$ -extensions, *Mat. Sbornik* **20** (1947), 351–363.